M321/L

THE OPEN UNIVERSITY

Third Level Course Examination 1977

PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

Wednesday, 26th October, 1977 **10.00 a.m. – 1.00 p.m.**

Time allowed: 3 hours

This examination comprises TWO parts.

You should attempt ALL the questions from Part I, and **not more than** FOUR questions from Part II. Part I carries 40% of the total marks for the examination. Each question in Part II carries 15% of the total marks.

You may answer questions in any order, writing your answers in the answer book(s) provided.

Use a separate answer book for each part. At the end of the examination, remember to write your name, student serial number and examination number on each answer book used. **Failure to do so will mean that your papers cannot be identified.**

PART I *This part carries 40% of the marks for the examination.*

Answer **ALL** *the eight questions in this part. Not all the questions in this part carry equal marks.*

Question 1 If $F(x, y)$ is a function defined on the (x, y)-plane, and (u, v) is a coordinate system in this plane defined by

$$x = 3u^2 + v^2 \quad \text{and} \quad y = -uv ,$$

express $\dfrac{\partial F}{\partial x}$ in terms of u, v, $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$.

Question 2 Laplace's equation in two-dimensional polar coordinates (r, θ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (r > 0, \text{ all } \theta) .$$

The method of separation of variables yields as a solution

$$u = (Ar^k + Br^{-k})(C\cos k\theta + D\sin k\theta) ,$$

where A, B, C, D and k are constants, and $k > 0$.

Write down:

(i) the most general solution of this type for

$$0 < r \leqslant a, \quad \text{all } \theta$$

satisfying

$$u(a, \theta) = 0 \qquad (\text{all } \theta)$$

$$u(r, \theta) = u(r, \theta + 2\pi) \qquad (0 < r \leqslant a) ,$$

where a is a positive constant;

(ii) the most general solution for $r \geqslant a$, $0 \leqslant \theta \leqslant \pi$ satisfying

$$\lim_{r\to\infty} u(r, \theta) = 0 \qquad (0 \leqslant \theta \leqslant \pi)$$

$$\frac{\partial u}{\partial n}(r, 0) = \frac{\partial u}{\partial n}(r, \pi) = 0 \qquad (r \geqslant a) .$$

Question 3 If S is a surface in 3-dimensional space bounding a region R, and v is a function that is harmonic in R and satisfies $\dfrac{\partial v}{\partial n} + kv = 0$ on S, where k is a positive constant, use the Divergence Theorem to prove that $v = 0$ in R and on S.

You may use the result

$$\operatorname{div}(u\,\mathbf{grad}\,v) = u\nabla^2 v + (\mathbf{grad}\,u)\cdot(\mathbf{grad}\,v) .$$

Question 4 The explicit scheme

$$\frac{1}{k}\Delta_t u_{i,j} = \frac{2}{h^2}\delta_x^2 u_{i,j} + u_{i,j}$$

is to be used to obtain an approximate solution of the parabolic equation

$$\frac{\partial U(x, t)}{\partial t} = 2\frac{\partial^2 U(x, t)}{\partial x^2} + U(x, t) .$$

Show that the order of the local truncation error is $O(h^2) + O(k)$, where h and k are the step sizes used for x and t respectively.

Question 5 Consider the problem

$$\frac{d^2u(x)}{dx^2} - (q\cos x - \lambda)\,u(x) = 0 \qquad \left(0 < x < \frac{\pi}{2}\right)$$

$$u(0) = u\left(\frac{\pi}{2}\right) = 0\ ,$$

where q is a positive constant and λ is an eigenvalue. Using a comparison theorem, show that the lowest eigenvalue lies between 4 and $4 + q$.

Question 6 (i) Which of the following equations has two distinct families of characteristics?

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial t^2} + u = 0 \qquad (1)$$

$$\frac{\partial^2 u}{\partial x^2} - 4x\frac{\partial^2 u}{\partial x\partial t} + 4x^2\frac{\partial^2 u}{\partial t^2} = 0 \qquad (2)$$

(ii) Consider a problem with initial data on the line $t = 0$ for the equation you chose in part (i). Draw a sketch showing the domain of dependence of the point where $x = 1$, $t = 3$, giving the equations of any lines or curves you draw.

Question 7 *Answer* either 7(*a*) or 7(*b*). *Do* NOT *attempt both.*

Either 7(a) If $u(x)$ satisfies the differential equation

$$e^{2x}\frac{d^2u(x)}{dx^2} + u(x) = 0 \qquad (x > 0)\ , \qquad (1)$$

show by putting $t = e^{-x}$ that the corresponding function $u(t)$ satisfies

$$t\frac{d^2u(t)}{dt^2} + \frac{du(t)}{dt} + tu(t) = 0 \qquad (0 < t < 1)\ . \qquad (2)$$

Hence find the solution of equation (1), given that $u(x)$ has the limit 2 for large x.

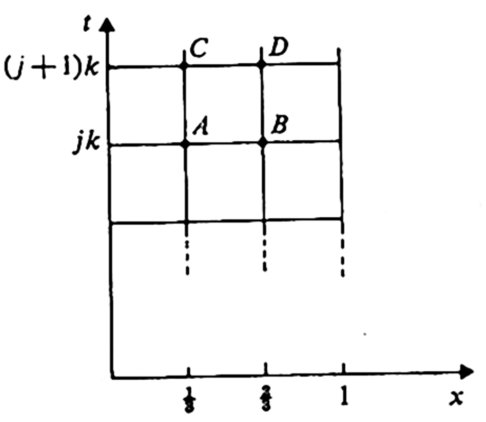
Or 7 (b) A finite-difference scheme for the equation

$$\frac{\partial U(x,t)}{\partial t} = U(x,t)^2\,\frac{\partial^2 U(x,t)}{\partial t^2}$$

is

$$u_{i,j+1} - u_{i,j} = r(u_{i,j+1})^2\,(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1})\ ,$$

where $x = ih$, $t = jk$, and $r = k/h^2$. It is required to use this scheme to solve the equation with the boundary condition $U(x, t) = 0$ when $x = 0$ and when $x = 1$, using a mesh with $h = \frac{1}{3}$. (See diagram.)



The values of u at the points A, B are supposed known (call them u_A, u_B).

(i) Write down the pair of equations which must be solved to find u_C and u_D.

(ii) Show how to solve these equations by Newton's iterative method, and comment on its convergence.

Turn over

Question 8 Briefly describe in your own words the concepts of *consistency*, *convergence*, and *stability* in connection with finite-difference approximations to differential equations.

PART II *This part carries 60% of the marks for the examination.*
Answer at most FOUR *questions from this part. All questions carry equal marks.*

Question 9

(i) In which parts of the region $\{(x, t): t \geqslant 0\}$ in the (x, t)-plane is the equation

$$t\frac{\partial^2 u}{\partial x^2} - x\frac{\partial^2 u}{\partial t^2} = 0 \qquad (t \geqslant 0)$$

elliptic, in which parts is it hyperbolic, and in which parts is it parabolic?

(ii) In the region (call it R) where this equation is hyperbolic, find the characteristics.

(iii) Briefly explain the importance of characteristics.

(iv) Find a coordinate system in R for which the equation takes its standard form.

(v) Carry out the transformation to standard form.

Question 10

An inflatable tent is made by attaching the edges of a rectangular sheet of suitable material to the ground and then pumping in air underneath. The height of the material above the ground at the point (x, y), which we denote by $u(x, y)$, satisfies the equation

$$\nabla^2 u = -F\,, \tag{1}$$

where F is a positive constant depending on the excess air pressure inside the tent. If the lengths of the sides of the rectangle are a and b, write down the boundary conditions for the equation (1) in a coordinate system whose axes are along two sides of the rectangle.

Write down a formula giving the volume of air inside the tent in terms of the function $u(x, y)$.

Using an extremum principle, write down an expression which gives a positive lower bound on this volume. Evaluate this lower bound for the case when $a = b$.

Question 11

Describe *one explicit* and *one implicit* finite-difference scheme for numerically solving the problem

$$\frac{\partial U(x,t)}{\partial t} = 3\frac{\partial^2 U(x,t)}{\partial x^2} \qquad (0 < x < 1, t > 0)$$

$$\left.\begin{array}{l} U(0,t) = 0 \\ U(1,t) = 0 \end{array}\right\} (t > 0)$$

$$U(x, 0) = f(x) \qquad (0 \leqslant x \leqslant 1)$$

where $f(x)$ is a given function.

Compare and contrast your two schemes with respect to stability, accuracy, and ease of computation, giving reasons (but not detailed proofs) for any assertions you make.

Question 12 (i) Consider the problem

$$\frac{\partial U(x,t)}{\partial t} = 4\frac{\partial^2 U(x,t)}{\partial x^2} \qquad (0 < x < 1, t > 0)$$

$$\left.\begin{aligned}\frac{\partial U(0,t)}{\partial x} &= 0\\ U(1,t) &= 0\end{aligned}\right\} \text{ for } t > 0$$

$$U(x,0) = f(x) \qquad (0 \leqslant x \leqslant 1)\ ,$$

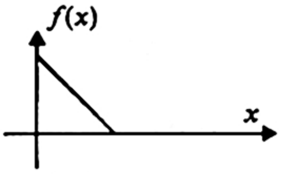
where f must satisfy suitable conditions but is otherwise arbitrary. Using the method of separation of variables, find the formal Fourier series representing $U(x, t)$ in terms of f.

(ii) Consider next the Fourier series found by you in part (i), evaluated at $t = 0$. Suppose the functions (*a*) and (*b*) below are each substituted for the initial function f in this series.

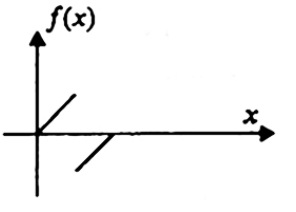
For each function state at which points x, with $0 \leqslant x \leqslant 1$, the resulting series does *not* converge pointwise to $U(x, 0)$. If there are such points, state the value of the sum of the series there.

Note carefully that no integrations are necessary in order to answer this question.

(*a*) $f(x) = 1 - x \qquad (0 \leqslant x \leqslant 1)$



(*b*) $f(x) = \begin{cases} x & (0 \leqslant x \leqslant \frac{1}{2}) \\ x - 1 & (\frac{1}{2} < x \leqslant 1) \end{cases}$



(iii) The function (*a*) of (ii) above satisfies $f'(x) = -1$ for all x in its domain, yet if the Fourier series for this function, as used in (i) above, is differentiated term by term, the resulting series sums to zero when $x = 0$. Explain briefly how the concept of uniform convergence helps to resolve this apparent contradiction.

Question 13 (i) Write in self-adjoint form the equation

$$\frac{d^2u(x)}{dx^2} + 2\alpha\frac{du(x)}{dx} + u(x) = F(x)\ , \tag{1}$$

where α is an unspecified number and F an unspecified real-valued function.

(ii) Show that if $|\alpha| < 1$ the influence function of the self-adjoint equation is given by

$$R(x,\xi) = \omega^{-1}\, e^{-\alpha(x+\xi)} \sin \omega(x - \xi)\ ,$$

where $\omega = \sqrt{(1 - \alpha^2)}$.

(iii) Obtain the solution of (1) which satisfies the conditions

$$u(0) = 0, \quad u'(0) = 0\ ,$$

expressing your solution in terms of R.

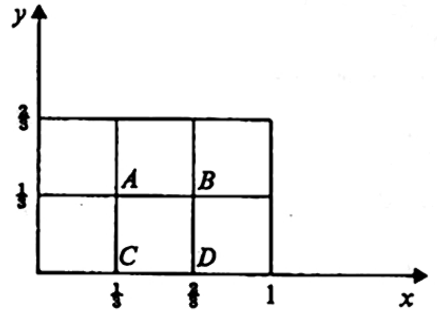
(iv) Show, if you have not already done so earlier in the question, that $R(x, \xi)$ and its derivatives have the properties characterizing the influence function, and that the solution obtained in (iii) is linear in F.

(v) Give a physical interpretation of $R(x, \xi)$.

Question 14 (i) Find a 5-point formula for the solution of

$$\frac{\partial^2 U(x, y)}{\partial x^2} + 2\frac{\partial^2 U(x, y)}{\partial y^2} = 0 \qquad (1)$$

on a square mesh of side h.



(ii) It is desired to solve equation (1) on the rectangle

$$0 < x < 1, \qquad 0 < y < \tfrac{2}{3},$$

with the boundary conditions

$$U(0, y) = U(1, y) = 0 \qquad (0 < y < \tfrac{2}{3})$$

$$\left.\begin{array}{l} U(x, \frac{2}{3}) = 0 \\ U(x, 0) - \dfrac{\partial U}{\partial y}(x, 0) = 1 \end{array}\right\} (0 < x < 1) .$$

Write down the system of equations for the approximate values of U at the four points marked A, B, C, D in the diagram, arising when the 5-point formula is used, with central differences for $\partial U/\partial y$, and $h = \frac{1}{3}$.

(iii) Write down the iteration matrix for Jacobi's method of solving this system of equations and state whether this method is convergent, giving the reason for your answer.

(iv) Is the S.O.R. method convergent for this system of equations? Give your reason.

Question 15 Consider the Sturm–Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) - q(x)\,u(x) + \lambda\,\rho(x)\,u(x) = 0 \qquad (0 < x < 1)$$

$$u(0) = u(1) = 0\ ,$$

where ρ and q are continuous, p is continuously differentiable, p and ρ are positive, and q is non-negative, for $0 \leqslant x \leqslant 1$.

(i) Show that, if $u(x)$ is a non-trivial solution of the integral equation

$$u(x) = \lambda \int_0^1 G(x, \xi)\,\rho(\xi)\,u(\xi)\,d\xi\ ,$$

where G is the Green's function for the problem

$$\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) - q(x)\,u(x) = -F(x)$$

$$u(0) = u(1) = 0\ ,$$

then $u(x)$ is an eigenfunction, and λ is an eigenvalue, of the Sturm–Liouville problem. Show all steps in your argument and state what theorems you are using.

(ii) Prove, stating what theorems you are using, that

$$\int_0^1\int_0^1 G(x, \xi)^2\,\rho(x)\,\rho(\xi)\,dx\,d\xi = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\ ,$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the Sturm–Liouville problem.

Question 16 A non-uniform string is stretched with tension T between the points $x = 1$, $x = 2$ on the x-axis of a rectangular coordinate system, both ends being held fixed. The density of the string at the point x is m/x^2, where m is a positive constant.

(i) Find the differential equation and boundary conditions that model small transverse vibrations of the string, if gravity is ignored.

(ii) Show that your differential equation has solutions of the form

$$e^{i\omega t} f(x)\ ,$$

where f satisfies

$$x^2\frac{d^2f(x)}{dx^2} + \frac{m\omega^2}{T}f(x) = 0\ .$$

(iii) By means of the substitution $x = e^z$, or otherwise, show that the angular frequencies, ω, are given by

$$\frac{4m\omega^2}{T} = 1 + \left(\frac{2n\pi}{\ln 2}\right)^2 \qquad (n = 1, 2, \ldots)\ .$$